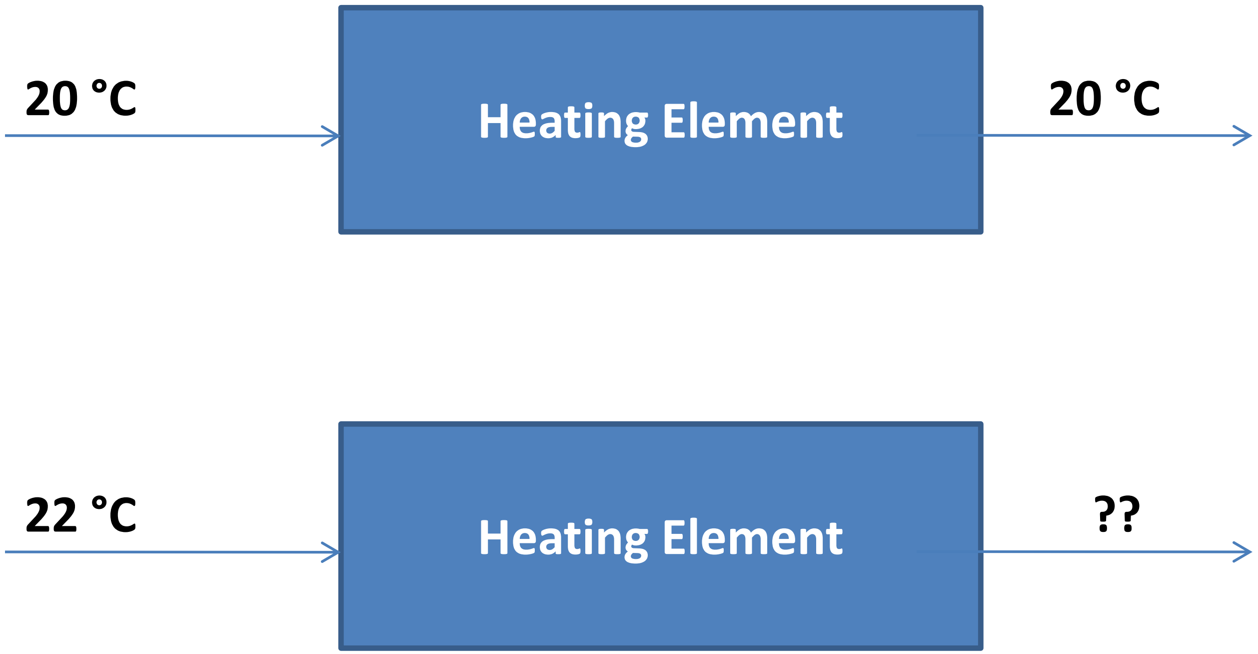
IPE 301
Measurement, Instrumentation and Control

Chapter 9
Transfer Function

Introduction



Static Response and Dynamic Response

- The term static response is used to describe the response of a system to an input without any reference to the time taken to reach that response.

  We are concerned with only what temperature the room ends up at.

- The term dynamic response is used when we also consider how it varies with time.

  We are concerned with not only what temperature the room ends up at but how it varies with time as it changes to the required value.

Static Response

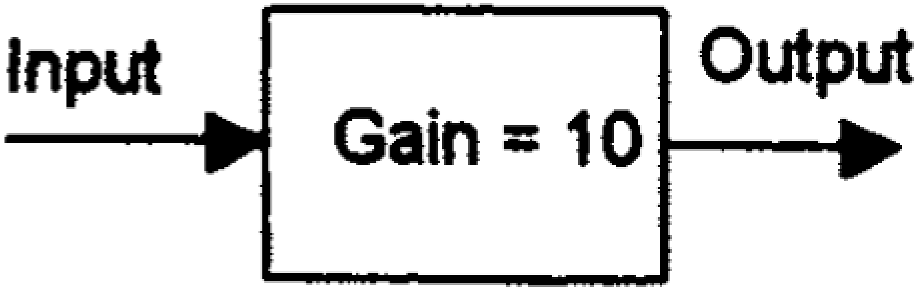
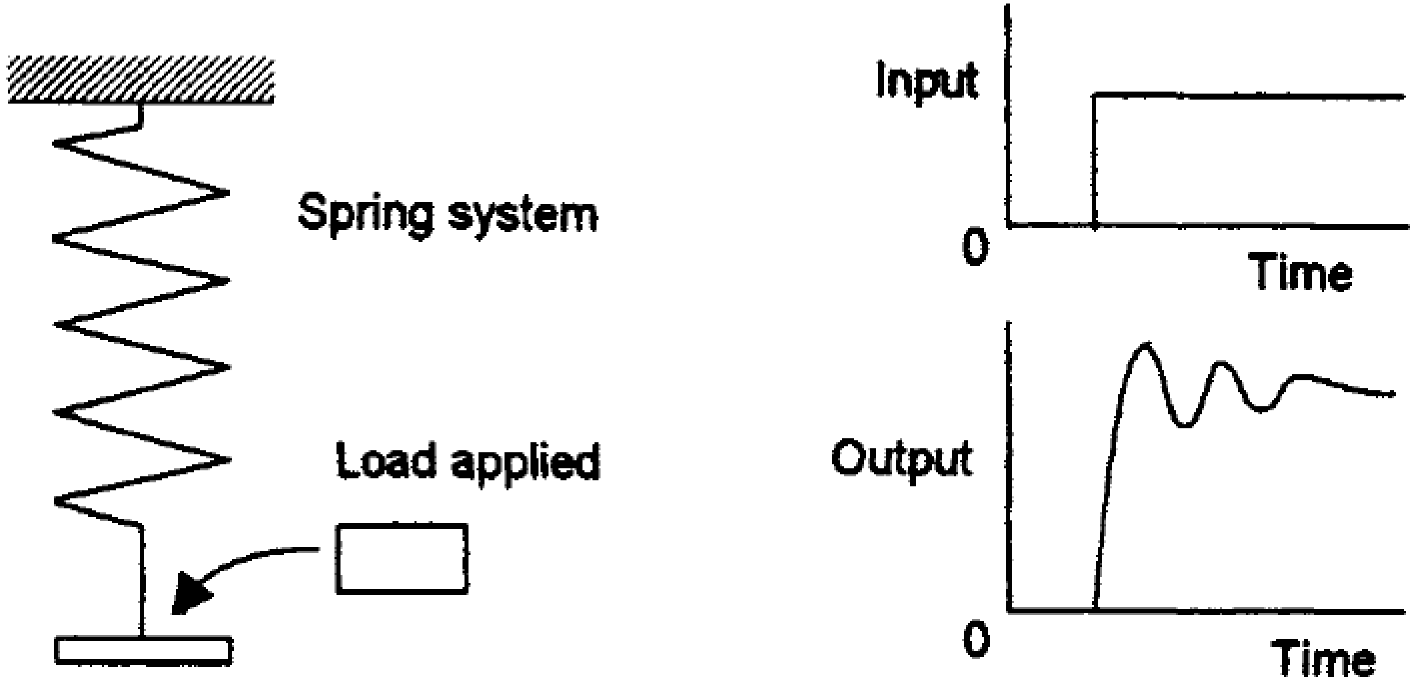


Fig: Amplifier system with the output ten times the input

Dynamic Response



Dynamic Response

- In a circuit with capacitance and resistance, when the voltage is switched on, i.e. there is a step voltage input to the system, then the current changes with time before eventually settling down to a steady value.
- With a temperature control system, when the thermostat is changed from 20°C to 22°C, the output does not immediately become 22°C but there is a change with time and eventually it may become 22°C.
- In general, the mathematical model describing the relationship between input and output for a system is likely to involve terms which give values which change with time and are described by a differential equation.

Gain

- For a system where the output is directly proportional to the input, we can write:

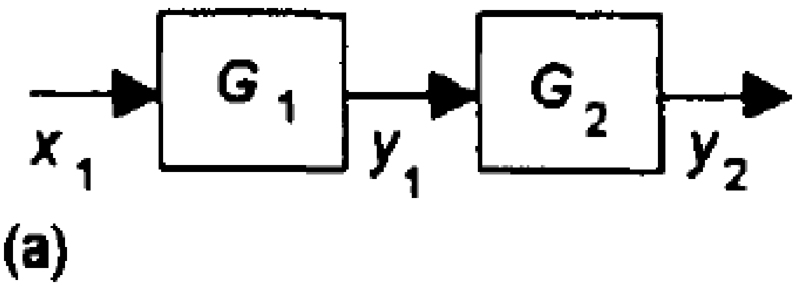
$$\text{Output} = G \times \text{Input}$$

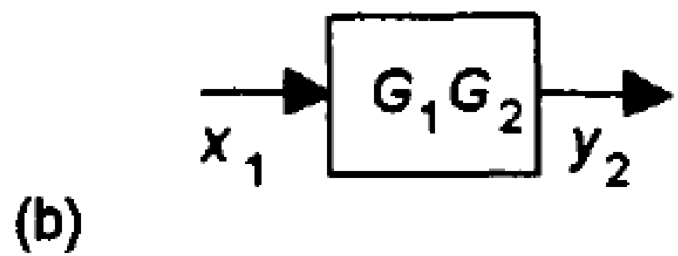
- Here, G = gain
- A motor has an output speed which is directly proportional to the voltage applied to its armature. If the output is 5 rev/s when the input voltage is 2 V, what is the system gain?

Solution: 5 rev/s = G × 2 V

So, G = (5 rev/s) ÷ 2 V = 2.5 (rev/s)/V

Gain of Systems in Series



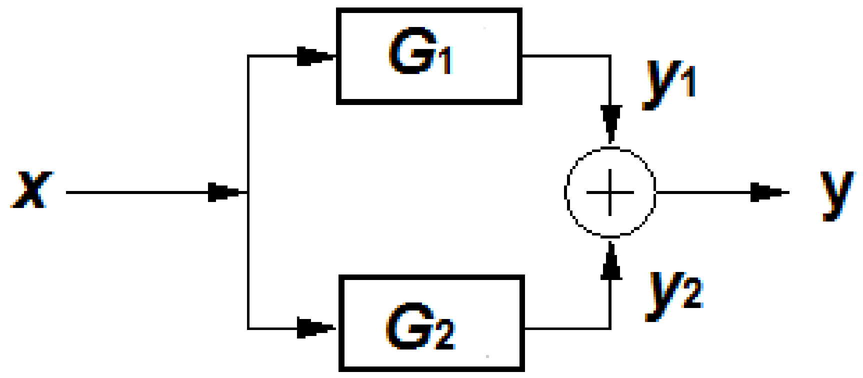


- ❖ (a) and (b) are equivalent
- ❖ For series-connected systems, the overall gain is the product of the gains of the constituent systems.

Gain of Systems in Series

Problem: **A system consists of an amplifier with a gain of 10 providing the armature voltage for a motor which gives an output speed which is proportional to the armature voltage, the constant of proportionality being 5 (rev/s)/V. What is the relationship between the input voltage to the system and the output motor speed?**

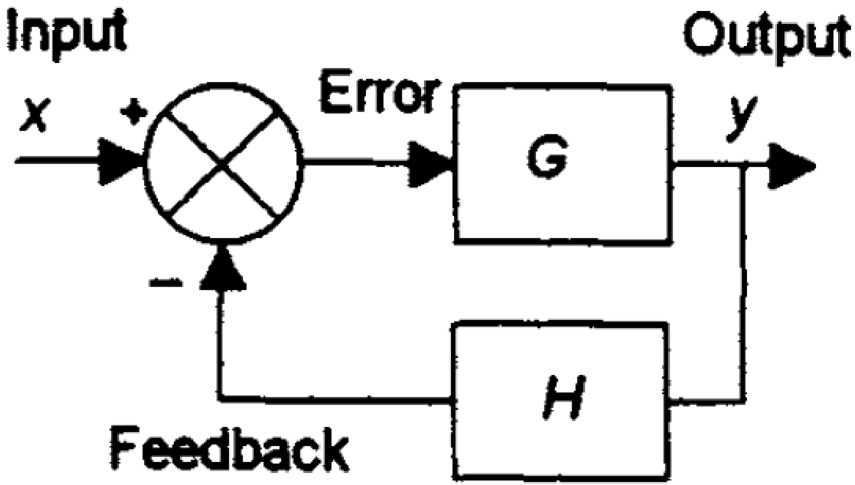
Gain of Systems in Parallel



❖ **What is the overall gain?**

Gain of Systems with Feedback Loop

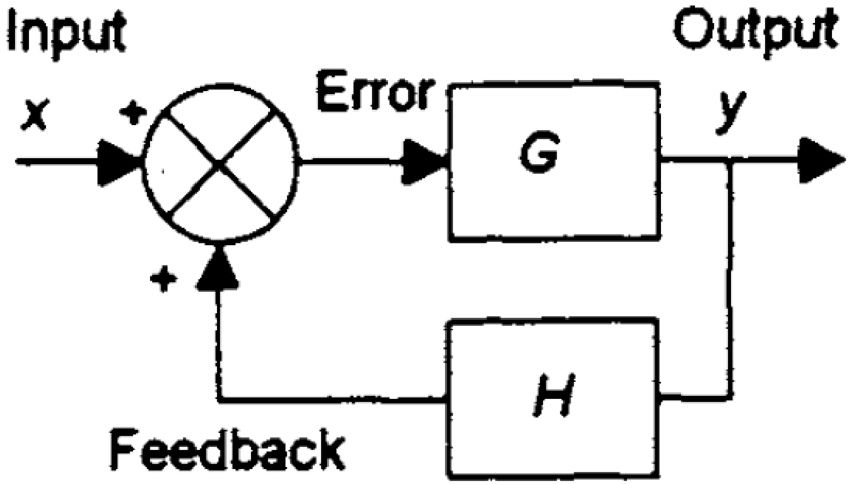
- **For system with negative feedback**



$$\text{system gain} = \frac{y}{x} = \frac{G}{1+GH}$$

Gain of Systems with Feedback Loop

- **For system with positive feedback**



$$\text{system gain} = \frac{y}{x} = \frac{G}{1 - GH}$$

Gain of Systems with Feedback Loop

- ***Problem:*** A negative feedback system has a forward path gain of 12 and a feedback path gain of 0.1. What is the overall gain of the system?

Introduction to Transfer Function

- **The relationship between the output and the input for elements used in control systems is frequently described by a differential equation.**
- **However, in order to make life simple, what we really need is a simpler relationship than a differential equation, even when the output varies with time.**
- **There is a way we can have such a simple form of relationship where the relationship involves time but it involves writing inputs and outputs in a different form.**
- **It is called the Laplace transform.**

Introduction to Transfer Function

- If we have a system composed of two elements in series with each having its input-output relationships described by a differential equation, it is not easy to see how the output of the system as a whole is related to its input.
- There is a way we can overcome this problem and that is to transform the differential equations into a more convenient form by using the Laplace transform.
- This form is a much more convenient way of describing the relationship than a differential equation since it can be easily manipulated by the basic rules of algebra.

Rules of Laplace Transform

- $v(t)$ when transformed becomes $V(s)$
- A constant k remains a constant k when transformed. Example: the voltage $3v(t)$ written as an s function is $3\,V(s)$.
- With no initial value at $t = 0$, dv/dt becomes $sV(s)$ when transformed. Example: with no initial values $4dv/dt$ as an s function is $4sV(s)$.

  Note: If there is an initial value v_o at $t = 0$ then the first derivative of a function of time dv/dt becomes $sV(s) - v_o$, i.e. we subtract any initial value, and $k dv/dt$ becomes $k[sV(s) - v_o]$.

  For example, if we have $v_o = 2$ at $t = 0$ then $4dv/dt$ becomes $4[sV(s) – 2]$.

Rules of Laplace Transform

- **With no initial values at $t = 0$, d^2v/dt^2 becomes $s^2V(s)$ when transformed. For example, with no initial values $4d^2v/dt^2$ as an s function is $4s^2V(s)$.**

  Note: If there are initial values v_o and $(dv/dt)_o$ then the second derivative of a function of time d^2v/dt^2 becomes $s^2V(s) - sv_o - (dv/dt)_o$ and kd^2v/dt^2 becomes $k\,[s^2V(s) - sv_o - (dv/dt)_o]$. For example, with initial values of $v_o = 2$ and $(dv/dt)_o = 3$ at time $t = 0$, then $4d^2v/dt^2$ as an s function is $4\,[s^2V(s) - 2s - 3]$

- $\int_0^t v\,dt$ becomes $\frac{1}{s}V(s)$ $\int_0^t kv\,dt$ becomes $\frac{1}{s}kV(s)$

Rules of Laplace Transform

Problem: Determine the Laplace transform for the following equations where we have *v* and v_c as functions of time and no initial values.

(a) $$v = RC\frac{\mathrm{d}v_C}{\mathrm{d}t} + v_C$$

(b) $$v = LC\frac{\mathrm{d}^2 v_C}{\mathrm{d}t^2} + RC\frac{\mathrm{d}v_C}{\mathrm{d}t} + v_C$$

Answers:

$$(a)\, V(s) = RCsV_c(s) + V_c(s)$$

$$(b)\, V(s) = LCs^2V_c(s) + RCsV_c(s) + V_c(s)$$

Transfer Function

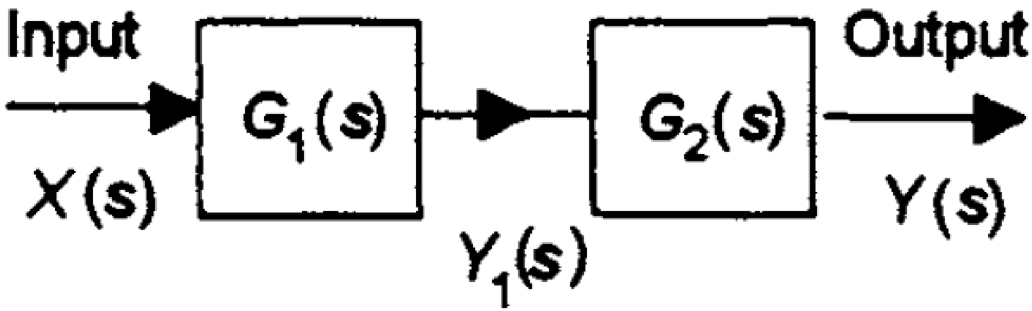
$$\text{Transfer Function, } G(s) = \frac{Y(s)}{X(s)}$$

Problem: Determine the transfer function for the mechanical system, having mass, stiffness and damping, and input *F* and output *y* and described by the differential equation:

$$F = m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky$$

System Transfer Functions

Systems in Series

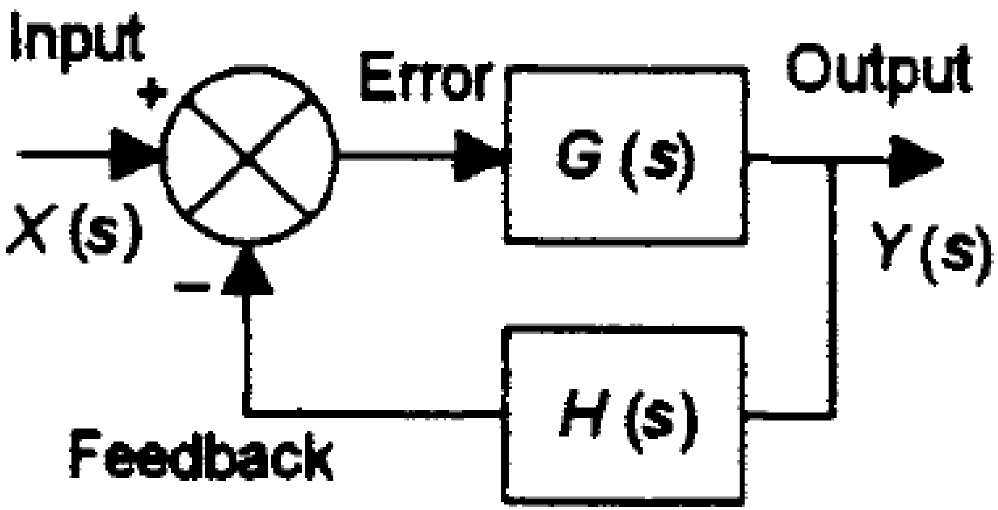


$$G_{\text{overall}}(s) = G_1(s) \times G_2(s)$$

Problem: **Determine the overall transfer function for a system which consists of two elements in series, one having a transfer function of 1/(s+1) and the other 1/(s+2).**

System Transfer Functions

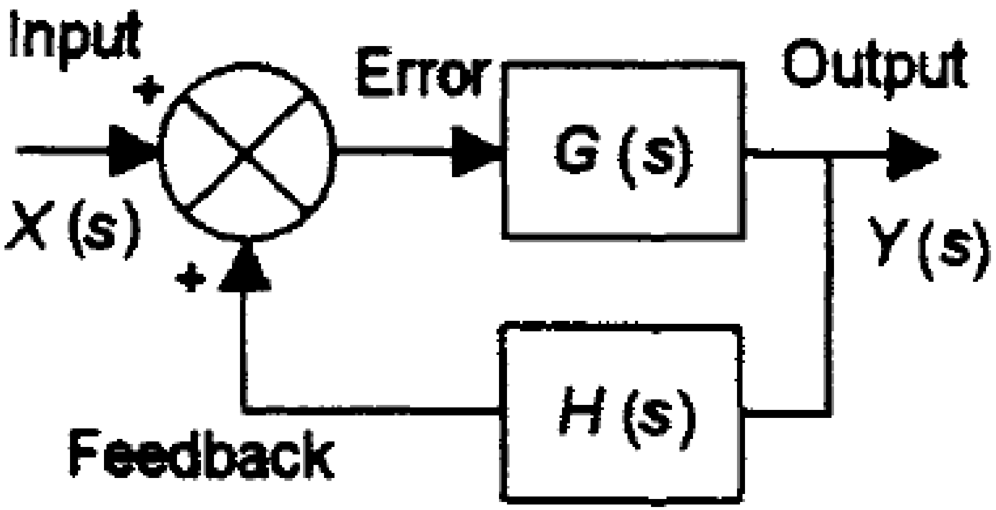
Systems with Feedback (Negative Feedback)



$$G_{\text{overall}}(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s) \times H(s)}$$

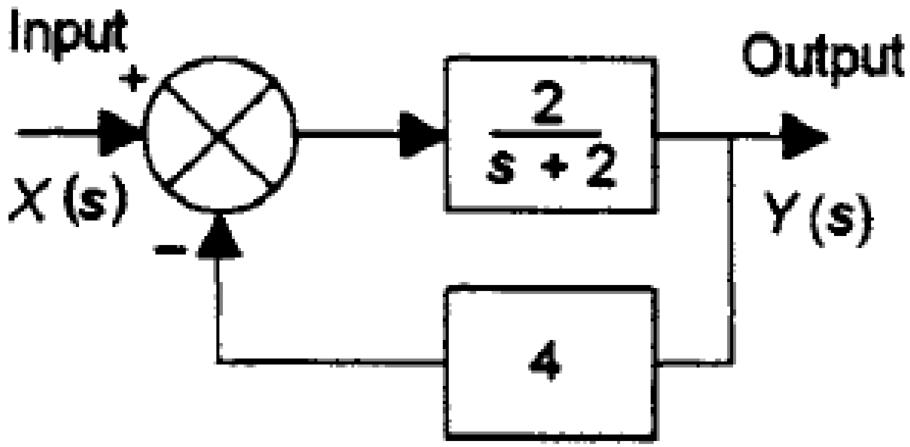
System Transfer Functions

Systems with Feedback (Positive Feedback)

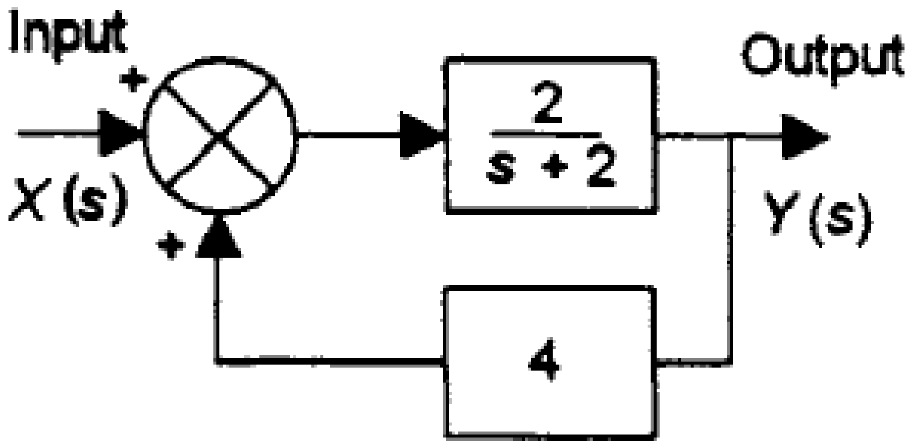


$$G_{\text{overall}}(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 - G(s) \times H(s)}$$

System Transfer Functions



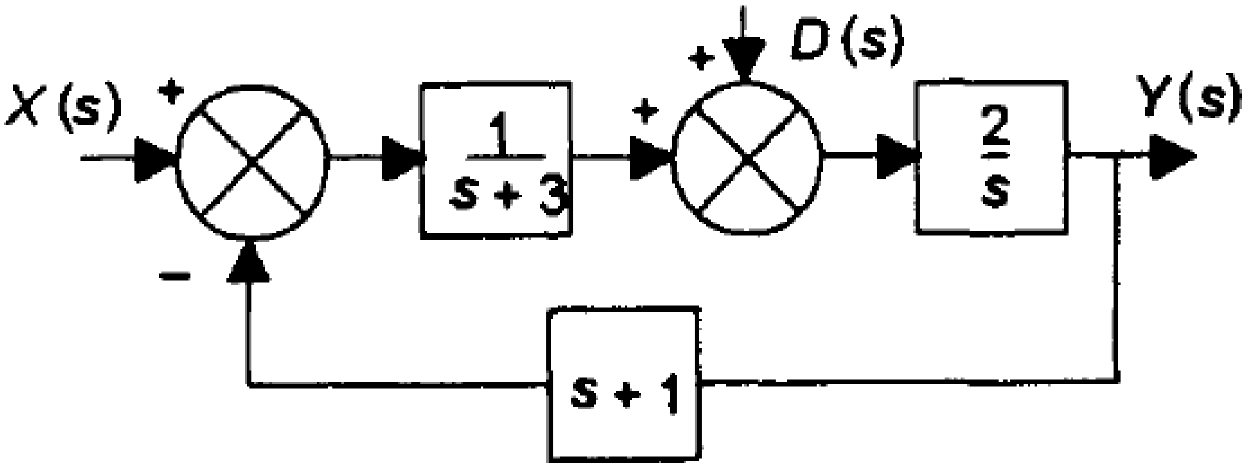
$$G_{\text{overall}}(s) = \frac{2}{s+10}$$



$$G_{\text{overall}}(s) = \frac{2}{s-6}$$

System Transfer Functions

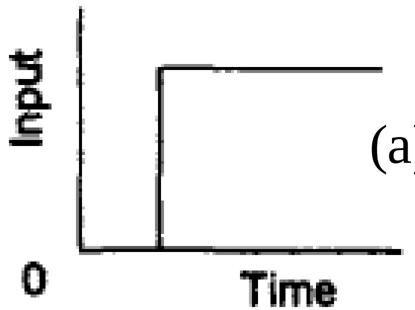
Multiple Inputs



$$Y(s) = \frac{2}{s(s+3)+2(s+1)}X(s)$$

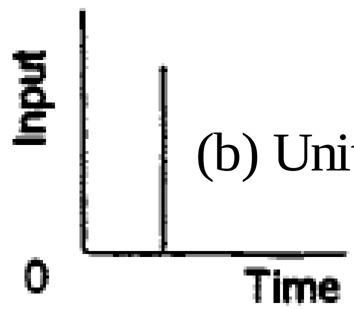
$$+\frac{2(s+3)}{s(s+3)+2(s+1)}D(s)$$

Forms of Input



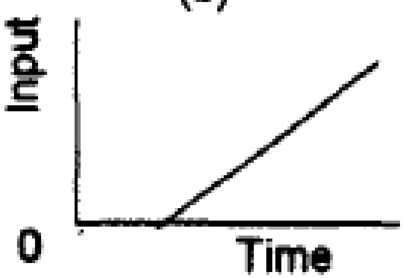
(a)

$$\text{(a) Unite Step Input Signal : Laplace Transform} = \frac{1}{s}$$



(b)

$$\text{(b) Unit Impulse Input Signal : Laplace Transform} = 1$$



(c)

$$\text{(c) Unit Ramp Signal : Laplace Transform} = \frac{1}{s^2}$$

Forms of Input

Problem: An electrical system has an input of a voltage of 2 V which is suddenly applied by a switch being closed. What is the Laplace transform of the input?

Problem: A controlled speed motor has a voltage input which is increased at the rate of 3 V per second. What is the Laplace transform of the input?

	Time function $f(t)$	Laplace transform $F(s)$
1	A unit impulse	1
2	A unit step	$\frac{1}{s}$
3	t, a unit ramp	$\frac{1}{s^2}$
4	e^{-at}, exponential decay	$\frac{1}{s+a}$
5	$1-e^{-at}$, exponential growth	$\frac{a}{s(s+a)}$
6	te^{-at}	$\frac{1}{(s+a)^2}$
7	$t-\frac{1-e^{-at}}{a}$	$\frac{a}{s^2(s+a)}$
8	$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
9	$(1-at)e^{-at}$	$\frac{s}{(s+a)^2}$
10	$1-\frac{b}{b-a}e^{-at}+\frac{a}{b-a}e^{-bt}$	$\frac{ab}{s(s+a)(s+b)}$
11	$\frac{e^{-at}}{(b-a)(c-a)}+\frac{e^{-bt}}{(c-a)(a-b)}+\frac{e^{-ct}}{(a-c)(b-c)}$	$\frac{1}{(s+a)(s+b)(s+c)}$

Inverse Laplace Transform

Problem: A system gives an output of $1/(s+5)$. What is the output as a function of time?

Problem: A system gives an output of $10/s(s+5)$. What is the output as a function of time?

Problem: A system has a transfer function of $1/(s + 2)$. What will be its output as a function of time when it is subject to a step input of 5V?

Problem: A system has a transfer function of $4/(s + 2)$. What will be its output as a function of time when subject to a ramp input of 10 V/s?

First Order System

$$\tau \frac{dy}{dt} + y = kx$$

$$\tau = \text{Time constant}$$

$$k = \text{Steady-state gain}$$

$$y = \text{Output};\ x = \text{Input}$$

What is the Laplace Transform of the upper equation?

$$\tau s Y(s) + Y(s) = kX(s)$$

$$G(s) = \frac{k}{\tau s + 1}$$

First Order System

When a first-order system is subject to a unit impulse input then $X(s) = 1$ and the output transform $Y(s)$ is:

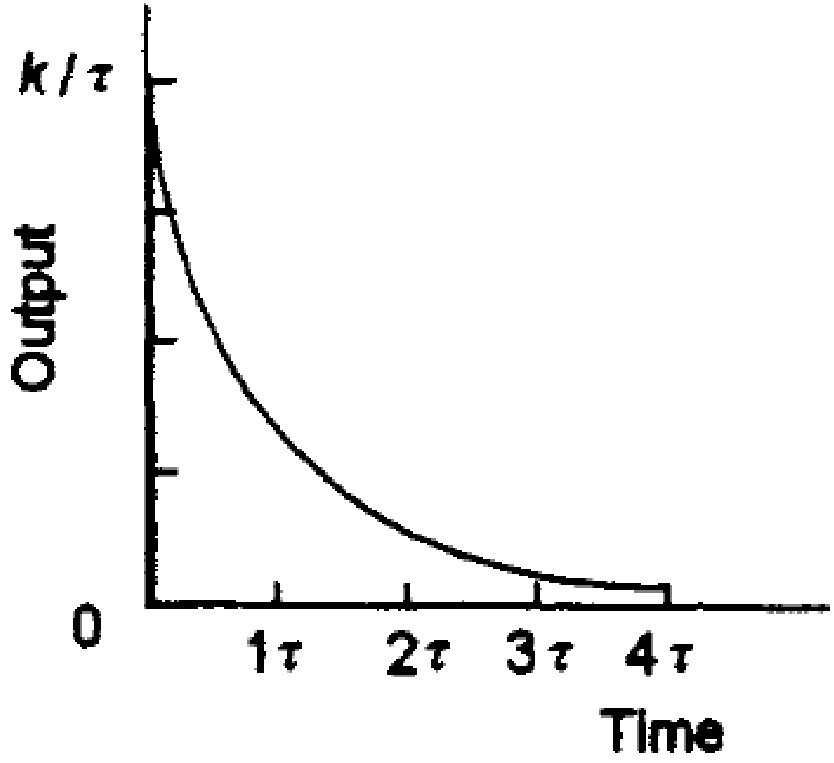
$$Y(s) = G(s) \times X(s) = \frac{k}{\tau s + 1} \times 1 = \frac{k}{\tau s + 1} = k \frac{1}{\tau} \frac{1}{s + \frac{1}{\tau}}$$

Performing inverse Laplace transform,

$$y(t) = k \, {}^{1}\!/_{\tau} \, e^{-\frac{1}{\tau}t}$$

First Order System

$$y(t) = k\,{}^{1}\!/_{\tau}\, e^{-\frac{1}{\tau}t}$$



Output with a unit impulse input to a first-order system

First Order System

When a first-order system is subject to a unit step input then $X(s) = 1/s$ and the output transform $Y(s)$ is:

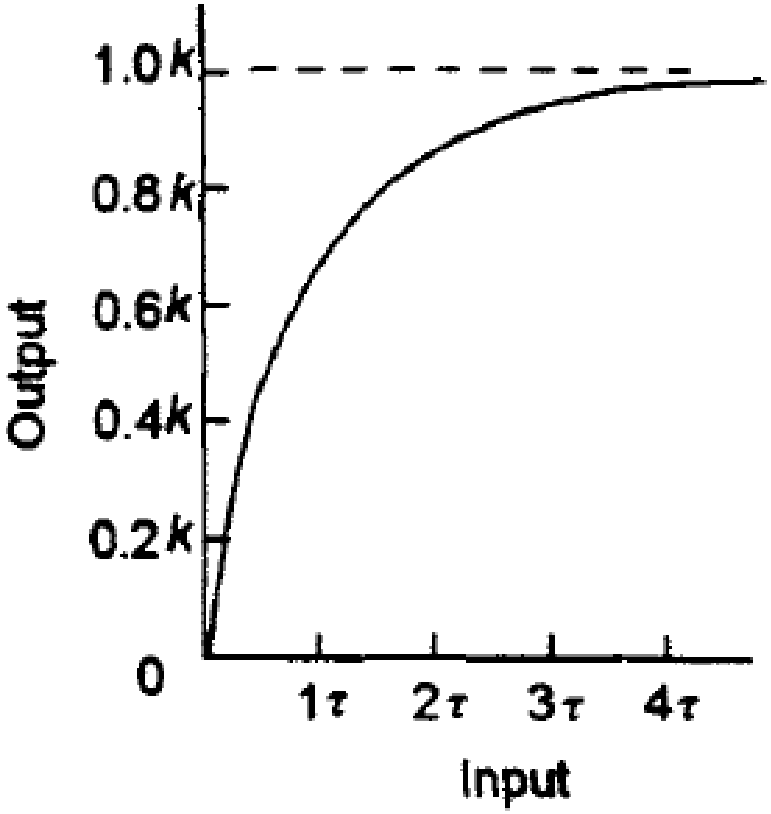
$$Y(s) = G(s) \times X(s) = \frac{k}{\tau s + 1} \times \frac{1}{s} = \frac{k}{s(\tau s + 1)} = k\frac{1/\tau}{s(s + 1/\tau)}$$

Performing inverse Laplace transform,

$$y(t) = k\left(1 - e^{-\frac{1}{\tau}t}\right)$$

First Order System

$$y(t) = k\left(1 - e^{-\frac{1}{\tau}t}\right)$$



Behaviour of a first-order system when subject to a unit step input

First Order System

Problem: A circuit has a resistance *R* in series with a capacitance *C*. The differential equation relating the input *v* and output v_c, i.e. the voltage across the capacitor, is:

$$v = RC\frac{dv_c}{dt} + v_c$$

Determine the output of the system when there is a 3 V impulse input.

First Order System

Problem: A thermocouple which has a transfer function linking its voltage output *V* and temperature input of:

$$G(s) = \frac{30 \times 10^{-6}}{10s + 1} V / °C$$

Determine the response of the system when it is suddenly immersed in a water bath at 100 °C.

First Order System Parameters

Time Constant: The time constant for a first-order system when subject to a step input is the time taken for the output to reach 0.63, of the steady-state value.

Delay Time: The delay time is the time required for the output response to reach 50% of its steady-state value.

Rise Time: The rise time, is the time required for the output to rise from 10% to 90% of its steady-state value.

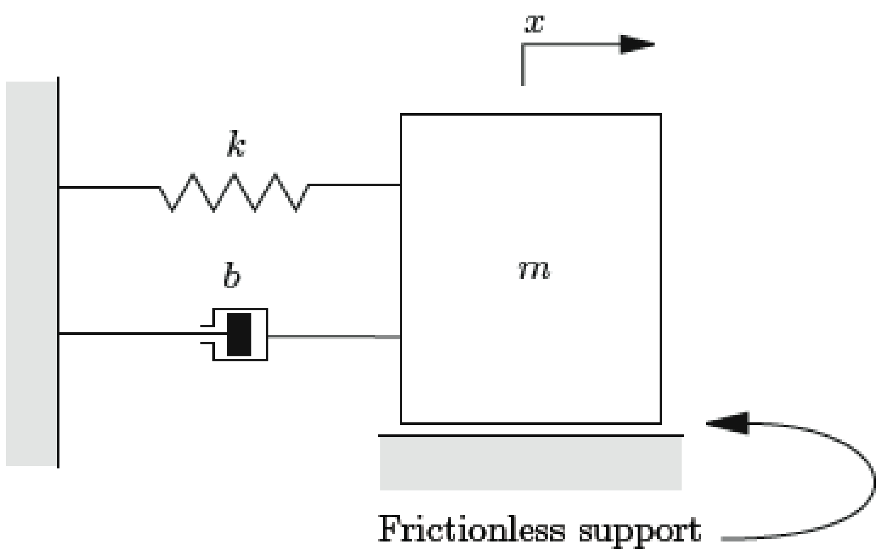
First Order System Parameters

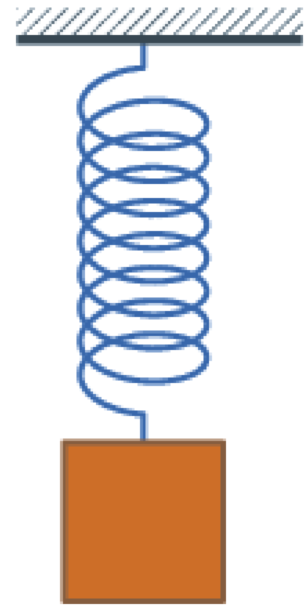
Problem: **Determine the time constant, delay time and the rise time for a first-order system with the transfer function:**

$$G(s) = \frac{3}{2s+1}$$

Problem: **A mercury-in-glass thermometer acts as a first-order system with an input of temperature and an output of the mercury position against a scale. The thermometer is initially at 0°C and is then suddenly placed in water at l00°C. After 80 s the thermometer reads 98°C. Determine (a) the time constant, (b) the delay time, (c) the rise time.**

Second Order Systems





Second Order Systems

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = k\omega_n^2 x$$

ω_n = natural angular frequency (rad/s)

ζ = damping coefficient

What is the Laplace Transform of the upper equation?

$$s^2Y(s) + 2\zeta\omega_n sY(s) + \omega_n^2 Y(s) = k\omega_n^2 X(s)$$

Transfer Function:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Second Order Systems

When a second-order system is subject to a **unit step** input then $X(s) = 1/s$ and the output transform $Y(s)$ is:

$$Y(s) = G(s) \times X(s)$$

$$= \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \times \frac{1}{s}$$

$$= \frac{k\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

Second Order Systems

The output *Y(s)* can be of three forms depending on the value of the damping coefficient.

The quadratic term in the denominator is:

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

To factorize it, we need to find its roots. Let p_1 and p_2 be the roots.

$$\therefore p = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Second Order Systems

Case 1: $\zeta > 1 \ \therefore p_1, p_2$ are real (overdamped system)

$$Y(s) = \frac{k\omega_n^2}{s(s+p_1)(s+p_2)}$$

$$y(t) = \frac{k\omega_n^2}{p_1 p_2}\left[1 - \frac{p_2}{p_2 - p_1}e^{-p_1 t} + \frac{p_1}{p_2 - p_1}e^{-p_2 t}\right]$$

Since $p_1 p_2 = \omega_n^2$

The steady-state value is k

Second Order Systems

Case 2: $\zeta = 1$ $\quad \therefore p_1, p_2$ are real and equal (critically damped system)

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$\therefore p_1 = p_2 = -\omega_n$$

$$Y(s) = \frac{k\omega_n^2}{s(s+\omega_n)^2}$$

$$Y(s) = k\left[\frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}\right]$$

$$y(t) = k\left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right]$$

Second Order Systems

Case 3: $\zeta < 1, 0 < \zeta < 1$

$\therefore p_1, p_2$ are not real (underdamped system)

$$y(t) = k\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{\left(1-\zeta^2\right)}t + \phi\right)\right]$$

Here, $\cos\phi = \zeta$

The angular frequency of the underdamped oscillation is :

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

Second Order Systems

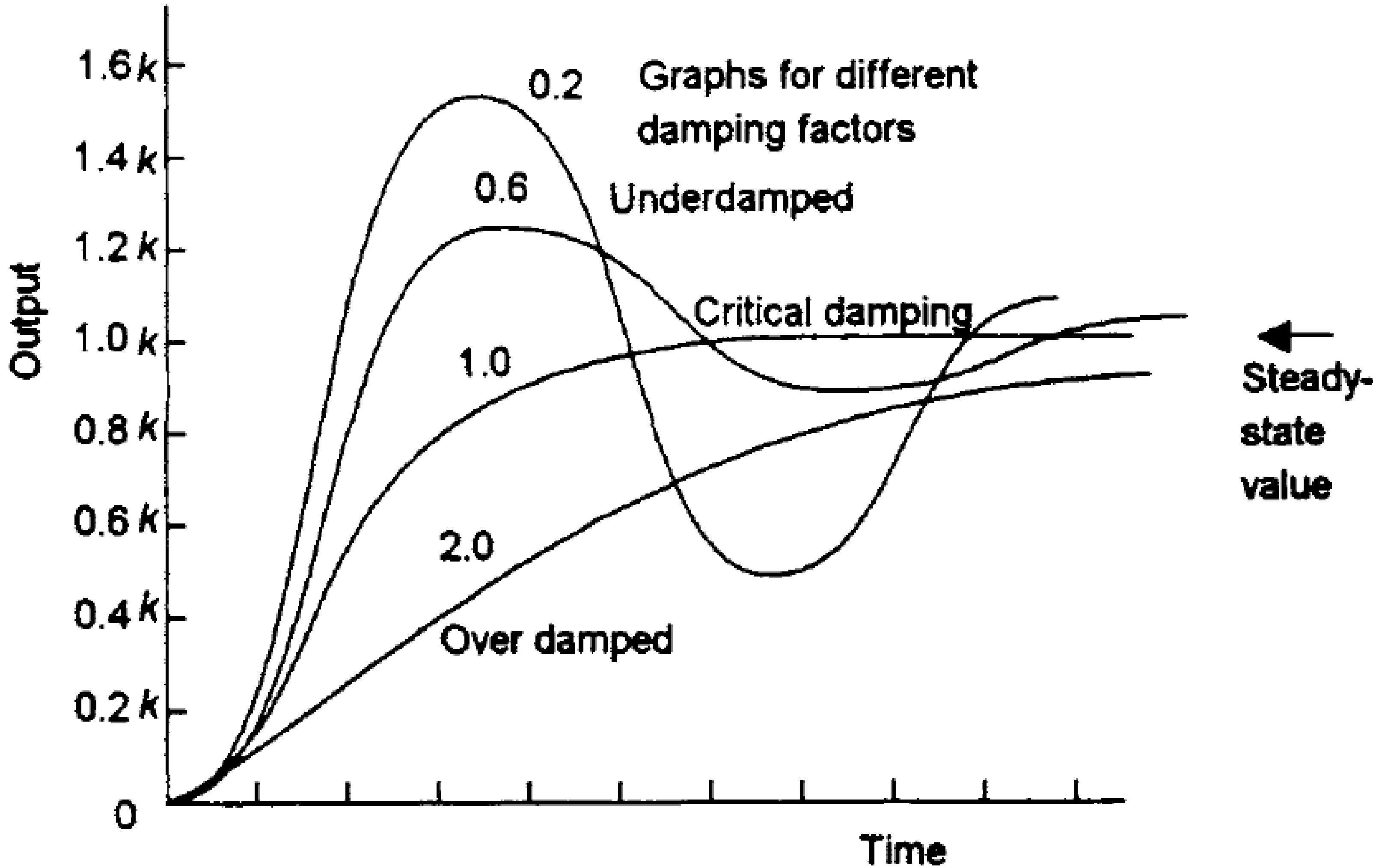
1 With *no damping*, i.e. $\zeta = 0$, the system output oscillates with a constant amplitude and a frequency of ω_n

2 With *underdamping* i.e. $\zeta < 1$, the output oscillates but the closer the damping factor is to 1 the faster the amplitude of the oscillations diminishes.

3 With *critical damping*, i.e. $\zeta = 1$, there are no oscillations and the output just gradually approaches the steady-state value.

4 With *overdamping*, i.e. $\zeta > 1$, the output takes longer than critical damping to reach the steady-state value.

Second Order Systems



Second Order System

Problem: What will be the state of damping of a system having the following transfer function and subject to a unit step input.

$$G(s) = \frac{1}{s^2 + 8s + 16}$$

Problem: A system has an output *y* related to the input *x* by the differential equation:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = x$$

What will be the output from the system when it is subject to a unit step input?